Exercise 9

Use power series to solve the differential equation.

$$y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution centered here.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Differentiate the series with respect to x once more.

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these formulas into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x inside the summand.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Because of n in the summand, the second series can start from $n = 0$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Make the substitution $n = k + 2$ in the first series and the substitution $n = k$ in the second and third series.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} x^{(k+2)-2} - \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

Simplify the first sum.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^{k} - \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

Now that all the sums start from $k = 0$ and have x^k in the summand, they can be combined.

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} x^{k} - k a_k x^k - a_k x^k \right] = 0$$

Simplify the summand.

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} - (k+1)a_k\right] x^k = 0$$

Since x^k isn't zero, the quantity in square brackets must be zero.

$$(k+2)(k+1)a_{k+2} - (k+1)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{1}{k+2} a_k$$

In order to determine a_k, plug in values for k and try to find a pattern.

$$\begin{aligned}
k = 0: &\quad a_2 = \frac{1}{0+2} a_0 = \frac{a_0}{2} \\
k = 1: &\quad a_3 = \frac{1}{1+2} a_1 = \frac{a_1}{3} \\
k = 2: &\quad a_4 = \frac{1}{2+2} a_2 = \frac{1}{4}\left(\frac{a_0}{2}\right) = \frac{a_0}{2\cdot 4} \\
k = 3: &\quad a_5 = \frac{1}{3+2} a_3 = \frac{1}{5}\left(\frac{a_1}{3}\right) = \frac{a_1}{3\cdot 5} \\
k = 4: &\quad a_6 = \frac{1}{4+2} a_4 = \frac{1}{6}\left(\frac{a_0}{2\cdot 4}\right) = \frac{a_0}{2\cdot 4\cdot 6} \\
k = 5: &\quad a_7 = \frac{1}{5+2} a_5 = \frac{1}{7}\left(\frac{a_1}{3\cdot 5}\right) = \frac{a_1}{3\cdot 5\cdot 7} \\
&\quad \vdots
\end{aligned}$$

The general formula for the even subscripts is

$$a_{2m} = \frac{a_0}{(2m)!!} = \frac{a_0}{2^m m!},$$

and the general formula for the odd subscripts is

$$a_{2m+1} = \frac{1}{(2m+1)!!} a_1 = \frac{2^m m!}{(2m+1)!} a_1.$$

Therefore, the general solution is

$$\begin{aligned}
y(x) &= \sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=0}^{\infty} a_{2m} x^{2m} + \sum_{m=0}^{\infty} a_{2m+1} x^{2m+1} \\
&= \sum_{m=0}^{\infty} \frac{a_0}{2^m m!} x^{2m} + \sum_{m=0}^{\infty} \frac{2^m m!}{(2m+1)!} a_1 x^{2m+1},
\end{aligned}$$

where a_0 and a_1 are arbitrary constants. Differentiate it with respect to x.

$$y'(x) = \sum_{m=1}^{\infty} \frac{a_0}{2^m m!}(2mx^{2m-1}) + \sum_{m=0}^{\infty} \frac{2^m m!}{(2m+1)!} a_1 (2m+1)x^{2m}$$

Apply the initial conditions to determine a_0 and a_1.

$$y(0) = a_0 = 1$$

$$y'(0) = a_1 = 0$$

Therefore,

$$y(x) = \sum_{m=0}^{\infty} \frac{1}{2^m m!} x^{2m}.$$